Tutorial 1 2022/9/21

1.1 Riemann integral

We first recall the definition of Riemann integrable:

Definition 1.1

A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a number I, such that for any $\epsilon > 0$, there exists $\delta > 0$, and for any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b, c = y_0 < y_1 < \cdots < y_m = d$ such that $\Delta x_i := x_{i+1} - x_i < \delta, \Delta y_j < \delta$ and for any tags $\tau = \{p_{ij}, p_{ij} \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]\}$, we have the Riemann sum

$$S(f, P, \tau) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(p_{ij}) \Delta x_i \Delta y_j$$

such that

$$|S(f, P, \tau) - I| < \epsilon.$$

♣

The definition adapts to arbitrary dimension by changing the number of parameters.

Problem 1.1 Let

$$\phi(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$$

show that $\phi(x)$ is not Riemann integrable on $[0, 1]$.

Proof Assume it is integrable, then we have $\exists I, \forall \epsilon > 0, \exists \delta > 0, \forall \mathcal{P}$ partition and $\forall \tau$ tags on $\mathcal{P}$, the Riemann sum $S(\phi, P, \tau) = \sum_{i=0}^{n-1} f(p_i) \Delta x_i$ satisfies the inequality

$$|S(f, P, \tau) - I| < \epsilon.$$

Then we have

$$\frac{1}{p_0} x_1 = f(p_0) \Delta x_0 < -\sum_{i=1}^{n-1} f(p_i) \Delta x_i + \epsilon + I.$$

This holds for any p_0 such that $0 < p_0 < x_1$. Let p_0 tend to 0, then $f(p_0)\Delta x_0$ can not be bounded as above, which is a contradiction.

Remark In fact, you can use the same argument to show that if a function is not bounded on some interval, then it is not integrable.

Problem 1.2 Let

$$g(x, y) = \begin{cases} 1 & x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

show that g is not Riemann integrable on $R = [a, b] \times [c, d]$.

Proof Let P be any partition of the rectangle. By choosing tags points $p_{ij} \left(x_i^*, y_j^*\right)$ where x_i^* and y_j^* are rational numbers,

$$\sum_{i,j} g\left(x_i^*, y_j^*\right) \Delta x_i \Delta y_j = \sum_{i,j} \Delta x_i \Delta y_j$$

which is equal to the area of R. On the other hand, by choosing the tags so that x_i^* is irrational, $g\left(x_i^*, y_j^*\right) = 0$

so that

$$\sum_{i,j} g\left(x_i^*, y_j^*\right) \Delta x_i \Delta y_j = \sum_{i,j} 0 \times \Delta x_i \Delta y_j = 0.$$

Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit no matter how small their norms are. We conclude that g is not integrable.

Remark You can conclude that

$$g_n(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \forall i, x_i \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable

1.2 Henstock–Kurzweil integral

We see that many functions are not integrable (For example as long as it is not bounded), but in many cases they would have calculable area below the graph, such as $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}\,x = 2$ where $\frac{1}{\sqrt{x}}$ is not integrable on $[0,1]$. Also we would expect the integration of g_n on any rectangle to be 0 as the size of the set of rational numbers is way too smaller than that of the set of irrational numbers.

One way to make an adjustment is by Lebesgue's theory on integration which you will learn in a course real analysis. Another simpler way is what I will show you below.

Definition 1.2

A function $f : [a,b] \to \mathbb{R}$ is HK-integrable on $[a,b]$ if there exists a number I, such that for any $\epsilon > 0$, there exists a positive function $\delta : [a,b] \to (0,\infty)$, and for any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$ and for any tags $\tau = \{p_i, p_i \in [x_i, x_{i+1}]\}$ such that $x_{i+1} - p_i < \delta(p_i)$ and $p_i - x_i < \delta(p_i)$, we have

$$|S(f, P, \tau) - I| < \epsilon.$$

♣

Note you can get this definition generalized to arbitrary dimension in the manner as we define the Riemann integrable in dimension two. The positive function δ is what the difference from Riemann integrable. In fact, a contant δ in the definition of Riemann integrable could be viewed as a function on $[a,b]$, so Riemann integrable imples HK-integrable.

Problem 1.3 Let

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

show that g is HK-integrable on $[0,1]$ with integration $I = 0$.

Proof We list all rational numbers in $[0,1]$ as $\{q_1, q_2, \cdots, q_n, \cdots\}$. For $\epsilon > 0$, we choose a delta

$$\delta(x) = \begin{cases} \frac{1}{2^{n+1}}\epsilon & x = q_n \\ 1/2 & \text{otherwise} \end{cases}$$

Consider the Riemann sum $S(g, P, \tau) = \sum g(p_i)\Delta x_i$. If p_i equals to some rational number q_n, then $\Delta x_i = x_{i+1} - x_i = x_{i+1} - q_n + q_n - x_i < 2\delta(q_n) = \frac{1}{2^n}\epsilon$. So we have

$$0 \le S(g, P, \tau) = \sum g(p_i)\Delta x_i \le \sum g(q_n)\frac{1}{2^n}\epsilon = \epsilon$$

Therefore the integration is 0.

Problem 1.4 Let

$$\phi(x) = \begin{cases} \frac{1}{\sqrt{x}} & x > 0 \\ 0 & x = 0 \end{cases}$$

show that $\phi(x)$ is HK-integrable on $[0,1]$.

Proof we shall show that $\int_0^1 f(x)dx = 2$. Let any $\varepsilon > 0$ be given; we are to exhibit a coresponding function δ. Let $\delta(0) = \frac{1}{16}\varepsilon^2$, and for each $x > 0$ choose $\delta(x) > 0$ small enough so that

$$[u,v] \subseteq (x-\delta(x), x+\delta(x)) \Longrightarrow \left|\frac{2}{\sqrt{u}+\sqrt{v}} - \frac{1}{\sqrt{x}}\right| < \frac{\varepsilon}{2}.$$

It follows easily that $\left|\left(2\sqrt{x_i} - 2\sqrt{x_{i-1}}\right) - f\left(p_{i-1}\right)\left(x_i - x_{i-1}\right)\right|$ is less than $\frac{1}{2}\left(x_i - x_{i-1}\right)\varepsilon$ when $p_{i-1} \neq 0$, or less than $\frac{1}{2}\varepsilon$ when $p_{i-1} = 0$. From this we obtain $\left|2 - \sum_{i=1}^N f\left(p_{i-1}\right)\left(x_i - x_{i-1}\right)\right| < \varepsilon$. The preceding argument proves the existence of a suitable δ, but it does not provide an explicit formula for δ. One choice that will work is

$$\delta(x) = \min\left\{\frac{1}{2}x, \frac{1}{4}x^{3/2}\varepsilon\right\} \qquad \text{when } x > 0.$$

To see that this will work, reason as follows: If $|x-u| < \delta(x)$, then $u > \frac{1}{2}$ x, hence

$$\sqrt{xu}(\sqrt{u}+\sqrt{x}) > x^{3/2}\sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}+1\right) > \frac{x^{3/2}}{2}$$

hence

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{u}}\right| = \left|\frac{u-x}{\sqrt{xu}(\sqrt{u}+\sqrt{x})}\right| < \frac{\delta(x)}{\frac{1}{2}x^{3/2}} \leq \frac{1}{2}\varepsilon.$$

Thus $\frac{1}{\sqrt{u}}$ lies in $\left(\frac{1}{\sqrt{x}} - \frac{\varepsilon}{2}, \frac{1}{\sqrt{x}} + \frac{\varepsilon}{2}\right)$. The same reasoning shows that $\frac{1}{\sqrt{v}}$ also lies in that interval, if $|x-v| < \delta(x)$. Finally, $\frac{2}{\sqrt{u}+\sqrt{v}}$ lies between $\frac{1}{\sqrt{u}}$ and $\frac{1}{\sqrt{v}}$, since $\frac{\sqrt{u}+\sqrt{v}}{2}$ lies between $\sqrt{u}$ and $\sqrt{v}$

Remark The intuition behind the definition of HK-integrable is to make the subinterval of the partition that contains a pathological tag as small as possible.